Game theory in a nutshell.

In general, game theory studies mathematical models of strategic interaction among decision makers (players).

Let's consider a game with n players $\{1, 2, \dots, n\}$, who have strategy profiles (accessible strategies) $\{S_1, \dots, S_n\}$ (each S_i is a set). Furthermore, let

$$u_1 : S_1 \times \dots \times S_n \to \mathbb{R}$$
$$u_2 : S_1 \times \dots \times S_n \to \mathbb{R}$$
$$\vdots$$
$$u_n : S_1 \times \dots \times S_n \to \mathbb{R}$$

be the collection of payoff functions of the players. In other words the value of $u_i(s_1, \dots, s_n)$ represents the outcome for player i in case player 1 chooses strategy $s_1 \in S_1$, player chooses $s_2 \in S_2$, etc.

One of the most important concepts is equilibrium (collection of strategies, that 'no player has initiative to deviate from'). The most famous and broadly used equilibrium is the one proposed by John Nash.

Notation. Let $s_{-i} := (s_1, s_2, \dots, s_{i-1}, s_{i+1}, \dots, s_n) \in S_1 \times \dots \times S_{i-1} \times S_{i+1} \times \dots \times S_n$ stand for a collection of strategies of all players except the i^{th}.

Def-n. (1) A collection of strategies $(s_1^*, s_2^*, \dots, s_n^*) \in S_1 \times S_2 \times \dots \times S_n$ is called a Nash equilibrium if

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \quad \forall i \in \{1, 2, \dots, n\}, \ \forall s_i \in S_i.$$

In other words, the i^{th} player would not be better off deviating from strategy s_i^* even provided the other players do not modify their strategies.

(2) $(s_1^*, s_2^*, \dots, s_n^*)$ is called a strong Nash equilibrium if for any subset $J = \{j_1, \dots, j_k\} \subset \{1, 2, \dots, n\}$

$$u_{j_s}(s_J^*, s_{-J}^*) \geq u_{j_s}(s_{j_1}, \dots, s_{j_k}, s_{-J}^*) \quad \text{for some } j_s \in J,$$

$$\forall (s_{j_1}, \dots, s_{j_k}) \in S_{j_1} \times \dots \times S_{j_k}$$

Here $s_J^* = (s_{j_1}^*, \dots, s_{j_k}^*)$ and $s_{-J}^* = (s_{i_1}^*, \dots, s_{i_{n-k}}^*)$ with $i_t \notin J$ (strategies of players not in J).

Rmk. The set of players in J is sometimes called a coalition.

Example. Consider an election with three candidates: A, B and C. Each voter's utility function needs to reflect his preferences. So if voter i has preferences

$$A \succ B \succ C, \text{ then } u_i(A)=1,\ u_i(B)=0,\ u_i(C)=-1,$$

for instance (we just need $u_i(A) > u_i(B) > u_i(C)$).

We will use the so called plurality rule, where the top candidate in the preference list of ~~a~~ a voter receives 1 point and the remaining two candidates none.

① Let's consider the case of 3 voters with preferences (A,B,C), (B,A,C) and (C,A,B), i.e.

$$u_1(A)=1,\ u_1(B)=0,\ u_1(C)=0, \text{ etc.}$$

Rmk. Here by $u_i(A)$, $u_i(B)$ and $u_i(C)$ we understand any collection of preferences that leads to A, B or C getting the most points.

Consider the following choices of strategies:

voter 1	A first	B first
voter 2	B first	B first
voter 3	A first	A first

First column is a NE
Second column is not:
voter 1 can pick A to be awarded a point and change the outcome to a more preferrable for him.

(2) 15 voters : 5 (B, A, C)

7 (A, B, C) - preferences.

3 (C, B, A)

Pick the strategies:

5 BAC	B
7 ABC	A
3 CBA	C.

Winner: A, 7 pts

Front-runner: B, 5 pts

3rd place: C, 3 pts.

Notice that for 3 voters with preferences (C, B, A) we get $u_i(B) > u_i(A)$. They can form a coalition and choose to vote for B, then B will get 8 points and win. This is an example of a Nash equilibrium, which is not a strong Nash equilibrium.

P+ epsilon attack.

Consider the following game. A question with two possible answers is given, e.g. 'Who won the last election?'. Each player picks one of the answers. If his answer coincides withe the one provided by majority of the players (>50%), the player receives a reward of P dollars, otherwise he gets nothing.

Clearly, there are two Nash equilibria. In our example, either everyone's response is 'Biden' or everyone's response is 'Trump'. Notice that it is likely that the incentive to give the 'true answer' (Biden) will prevail (unless there is a coordinated manipulation attempt).

Suppose an attacker makes the following offer: "if you choose Trump, but the majority chooses Biden, I will pay you P+1 dollar ($P+\mathcal{E}, \mathcal{E}>0$)'

Now the Nash equilibrium is for everyone to respond 'Trump'. This way the briber (attacker) manages to make the players pick the answer that he wants without paying a single penny!

Rmk. Recall that the 'concept of majority' (in terms of CPU) was crucial for the choice of continuation in bitcoin blockchain, so same strategy can be applied to bribe the nodes!